

**Probability generating functions (for discrete random variables)**

$$G_x(t) = \sum_x P(X = x)t^x$$

For a fair coin:

$$G_x(t) = \frac{1}{2} + \frac{1}{2}t$$

For the tossing of three coins, where  $X_1, X_2, X_3$  are independent:

$$Y = X_1 + X_2 + X_3$$

$$G_y(t) = \frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3 = \left(\frac{1}{2} + \frac{1}{2}t\right)^3$$

To find the probability of two heads

$$\begin{aligned} P(Y = 2) &= \frac{1}{2!} \cdot G_y^{(2)}(0) \\ &= \frac{1}{2} \left( \frac{3}{4} + \frac{3}{4}t \right) \Big|_{t=0} \\ &= \frac{1}{2} \cdot \frac{3}{4} \\ &= \frac{3}{8} \end{aligned}$$

Let  $X$  and  $Y$  be independent discrete random variables, and let  $Z = X + Y$ .

$$\begin{aligned} G_x(t) \cdot G_y(t) &= \left( \sum_x P(X = x)t^x \right) \cdot \left( \sum_y P(Y = y)t^y \right) \\ &= \sum_z \left( \sum_{x, y | x+y=z} P(X = x) \cdot P(Y = y) \right) t^z \\ &= \sum_z \left( \sum_{x, y | x+y=z} P(X = x \cap Y = y) \right) t^z \\ &= \sum_z P(Z = z)t^z \\ &= G_z(t) \end{aligned}$$

$$G_x(t) \cdot G_y(t) = G_{x+y}(t)$$

Considering calculus and probability generating functions

$$G_x(1) = \sum_x P(X = x) = 1$$

$$G'_x(1) = \sum_x xP(X = x)t^{x-1} \Big|_{t=1} = \sum_x xP(X = x) = E(x)$$

$$G''_x(1) = \sum_x x(x-1)P(X = x)t^{x-2} \Big|_{t=1} = \sum_x x(x-1)P(X = x)$$

$$\begin{aligned} \text{Var}(X) &= \sum_x x^2P(X = x) - (E(x))^2 \\ &= \sum_x x(x-1)P(X = x) + \sum_x xP(X = x) - \left( \sum_x P(X = x) \right)^2 \\ &= G''_x(1) + G'_x(1) - [G'_x(1)]^2 \end{aligned}$$

### The geometric distribution

This asks “how many attempts until a success?”

Let  $X$  be the first in a sequence of identical independent Bernoulli trials (two outcomes; success or failure) on which success occurs.

Let the probability of a success in a trial be  $\pi$ .

$$P(X = x) = (1 - \pi)^{x-1} \pi \mid x \in \mathbb{Z}^+.$$

A tennis player has a probability of  $\frac{1}{4}$  of playing a winning shot. What is the change has won the rally by the 6<sup>th</sup> shot?

$$P(X \leq 5) = 1 - P(X > 5)$$

$$= 1 - (1 - \pi)^5$$

$$= 1 - \left(\frac{3}{4}\right)^5$$

$$= 1 - \frac{243}{1024}$$

$$= \frac{781}{1024}$$

$$G_x(t) = \sum_{x=1}^{\infty} (1 - \pi)^{x-1} \pi t^x$$

$$= \pi t \sum_{x=1}^{\infty} [(1 - \pi)t]^{x-1}$$

$$= \frac{\pi t}{1 - (1 - \pi)t}$$

$$G'_x(t) = \frac{1 - (1 - \pi)t \pi - \pi t(-1 + \pi)}{(1 - (1 - \pi)t)^2} = \pi \left( \frac{1 - t + \pi t + t - t\pi}{(1 - (1 - \pi)t)^2} \right) = \frac{\pi}{(1 - (1 - \pi)t)^2}$$

$$\Rightarrow G'_x(1) = \frac{\pi}{(1 - (1 - \pi)1)^2} = \frac{1}{\pi}$$

$$G_x''(t) = \frac{0 + 2(1 - (1 - \pi)t)(1 - \pi)}{(1 - (1 - \pi)t)^4} = \frac{2\pi(1 - \pi)(1 - (1 - \pi)t)}{(1 - (1 - \pi)t)^4} = \frac{2\pi(1 - \pi)}{(1 - (1 - \pi)t)^3}$$

$$\Rightarrow G_x''(1) = \frac{2\pi(1 - \pi)}{\pi^3} = \frac{2 - 2\pi}{\pi^2}$$

$$\text{Var}(X) = G_x''(1) + G_x'(1) - [G_x'(1)]^2 = \frac{2 - 2\pi}{\pi^2} + \frac{1}{\pi} - \left[\frac{1}{\pi}\right]^2 = \frac{1 - \pi}{\pi^2}$$

The geometric distribution can be written as  $X \sim G(\pi)$

### The binomial distribution

How many successes in a sequence of identical independent Benoulli trials where the probability of success is  $\pi$  and there are  $\nu$  trials.

Consider  $X \sim B(\nu, \pi)$  [Benoulli distribution]

$$P(X = 0) = 1 - \pi$$

$$P(X = 1) = \pi$$

$$G_x(t) = (1 - \pi) + \pi t$$

Let  $Y = X_1 + X_2 + X_3 + \dots + X_\nu$  |  $X_i$  are independent.

$$G_y(t) = G_{x_1}(t) \cdot G_{x_2}(t) \cdot G_{x_3}(t) \dots G_\nu(t) = [G_x(t)]^\nu$$

$$= [(1 - \pi) + \pi t]^\nu$$

$$= \sum_{y=0}^{\nu} (1 - \pi)^{\nu-y} \pi^y \binom{\nu}{y} t^y$$

$$P(Y = y) = \binom{\nu}{y} (1 - \pi)^{\nu-y} \pi^y \quad |_{y \in \{0, 1, 2, \dots, \nu\}}$$

$$G_y(t) = [(1 - \pi) + \pi t]^\nu$$

$$G_y'(t) = \nu \pi [(1 - \pi) + \pi t]^{\nu-1}$$

$$\Rightarrow G_y'(1) = \nu \pi$$

$$G_y''(t) = \nu \pi^2 (\nu - 1) [(1 - \pi) + \pi t]^{\nu-2}$$

$$\Rightarrow G_y''(1) = \nu \pi^2 (\nu - 1)$$

$$\text{Var}(X) = G_x''(1) + G_x'(1) - [G_x'(1)]^2 = \nu \pi^2 (\nu - 1) + \nu \pi - \nu^2 \pi^2 = \nu \pi (1 - \pi)$$

The chance of a player winning a game is 0.4, what is the change of him winning 4 games in a match of 6 games?

$$\binom{6}{4} (0.6)^2 (0.4)^4 \approx 0.138$$

Or, calculated using values from the binomial distribution tables,  $n=6, x=4, p=0.4$

$$P(X = 4) = P(X \leq 4) - P(X \leq 3) = 0.9590 - 0.8208 \approx 0.138$$