More on probability generating functions

$$X \sim G(\pi) \Leftrightarrow G_x(t) = \frac{\pi t}{1-(1-\pi)t}$$

$$X \sim B(\nu,\pi) \Leftrightarrow G_x(t) = (1-\pi+\pi t)^\nu$$

Consider the behaviour as $\mu = \nu\pi$ is constant but $\nu \to \infty$.

$$\begin{aligned}\lim_{\nu\to\infty} G_x(t) &= \lim_{\nu\to\infty}(1-\pi+\pi t)^\nu \\ &= \lim_{\nu\to\infty}\left(1-\frac{\mu}{\nu}+\frac{\mu}{\nu}t\right)^\nu \\ &= \lim_{\nu\to\infty}\left(1+\frac{\mu(t-1)}{\nu}\right)^\nu\end{aligned}$$

Considering the limit of $\left(1+\frac{x}{n}\right)^n$ as $n \to \infty$ gives:

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(1+n\frac{x}{n}+\frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3+\ldots\right) \\ &= \lim_{n\to\infty}\left(1+x+\left(\frac{n-1}{n}\right)x^2+\ldots\right) \\ &= 1+x+\frac{x^2}{2}+\ldots \\ &= e^x.\end{aligned}$$

So:

$$\lim_{\nu\to\infty} G_x(t) = e^{\mu(t-1)}.$$

The limiting distribution has $G_x(t) = e^{\mu(t-1)}$. This is known as the Poisson distribution, $X \sim P(\mu)$.

$$G_x(t) = e^{\mu(t-1)} = e^{-\mu}e^{\mu t}$$

$$\sum_x P(X=x)t^x = e^{-\mu}\left(1+\mu t+\frac{\mu^2 t^2}{2!}+\frac{\mu^3 t^3}{3!}+\ldots\right)$$

$$\Rightarrow P(X=x) = \frac{\mu x}{x!}e^{-\mu} \mid x \in \mathbb{Z}_0^+.$$

Moment generating functions

$$M_x(t) = 1 + E(X)t + \left(\frac{E(X^2)t^2}{2!}\right) + \left(\frac{E(X^3)t^3}{3!}\right) + \ldots$$

$$\Rightarrow M_x'(0) = E(X)$$

$$\Rightarrow M_x''(0) = E(X^2) \text{ etc.}$$

$$Var(X) = E(X^2) - (E(X))^2 = M''(0) - [M'(0)]^2$$

$$M_x(t) = E(e^{Xt})$$

For the geometric distribution:

$$P(X = x) = (1-\pi)^{x-1}\pi$$

$$M_x(t) = \pi e^t + (1-\pi)\pi e^{2t} + (1-\pi)^2 \pi e^{3t} + \ldots = \frac{\pi e^t}{1 - e^t(1-\pi)}$$

…and the binomial distribution:

$$M_x(t) = (1 - \pi + \pi e^t)^\nu$$

Recalling the normal distribution, $N(0,1)$:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$E(g(x)) = \begin{cases} \sum_x P(X = x) g(x) \\ \int_x f(x) g(x)\, dx \end{cases}$$

$$M_z(t) = E(e^{zt})$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{zt} dz$$

$$= e^{-\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz$$

Substitute $\xi = z - t \Rightarrow \frac{d\xi}{dz} = 1$ gives:

$$M_z(t) = e^{-\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2} d\xi$$

$$= e^{\frac{1}{2}t^2}.$$